Eigenvalues and Eigenvectors - Problems 1

Find all the eigenvalues and corresponding eigenvectors, and say whether the matrix A can or cannot be diagonalized. If the matrix can be diagonalized, give a matrix P such that $P^{-1}AP = D$ is diagonal.

1.
$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

2.
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

3.
$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

4.
$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$$

Eigenvalues and Eigenvectors - Answers 1

1. The characteristic equation of A is $\det(A - \lambda I) = (\lambda - 5)(\lambda + 1) = 0$, so the eigenvalues are $\lambda = -1$ and $\lambda = 5$ with corresponding eigenvectors $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively.
Thus A can be diagonalised as
$$P^{-1}AP = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} = D.$$
Note that the order of the columns in P does not matter, provided that the order of the eigenvalues in D matches. In addition any non-zero multiples of the above eigenvectors also gives a correct diagonalisation.

2. The characteristic equation of A is $\det(A-\lambda I) = (1-\lambda)(2-\lambda)(3-\lambda) = 0$, so the eigenvalues are $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ 6 \\ 2 \end{pmatrix}$, respectively.
Thus A can be diagonalised as
$$P^{-1}AP = \begin{pmatrix} 1 & 2 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D.$$

3. The characteristic equation of A is $\det(A - \lambda I) = (1 - \lambda)(\lambda^2 - \lambda - 2) = 0$, so the eigenvalues are $\lambda = -1$, $\lambda = 1$ and $\lambda = 2$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$, respectively.
Thus A can be diagonalised as
$$P^{-1}AP = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D.$$

4. The characteristic equation of A is $\det(A - \lambda I) = (1 - \lambda)(\lambda^2 - \lambda + 4) = 0$, so the eigenvalues are $\lambda = 1$ and $\lambda = 2$, with $\lambda = 2$ being a repeated eigenvalue. The eigenvalue $\lambda = 1$ has corresponding eigenvector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. However there is only one independent eigenvector $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ corresponding to the eigenvalue $\lambda = 2$, so in this case A cannot be diagonalised.